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The flow in industrial cyclones

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A simple mathematical model for the flow in a conical cyclone is developed which allows solutions to be obtained in closed form. The flow in the main body of the cyclone is regarded as inviscid but the nature of the fluid entry to the device and the conical geometry ensure that secondary flows develop which make the flow highly rotational. The results of the theory are compared with data from two quite different experimental investigations, and good agreement is obtained.

1. Introduction

Over many years a great deal of effort has been devoted to obtaining a better working knowledge of industrial cyclones. The simple design of this device makes its principle of operation seem straightforward. Fluid containing particles of a different density, or even two different fluids, are injected tangentially at high speed into a vessel of circular cross-section. The centrifugal acceleration induced by the high rotational velocities causes the particles to move relative to the fluid. The high radial accelerations produced allow a relatively rapid migration of particles thus allowing a large rate of volume flow through the equipment.

The problem of assessing the collection of the distinct phases is not so straightforward as might at first appear. For example, in separating dense particles from a liquid of lower density the small-angled conical hydrocyclone would normally be used. The particles which are moving relative to the fluid enter the boundary layer of the conical wall and are then driven towards the underflow primarily by the component of the pressure gradient that acts towards the axis of the device. The flux of fluid and particles through this boundary layer is ejected through the underflow. The difficulty in assessing the performance of a cyclone, however, lies in establishing the likelihood of a particle entering this boundary layer. Although the particle moves relative to the fluid, there is, nevertheless, a component of the fluid velocity towards the axis of the cyclone which thus impedes the progress of the particle towards the boundary layer. Although these velocity components are very small they are certainly comparable with the particle drift velocity. It is therefore crucial that any model which attempts to analyse the separating efficiency of a cyclone must take account of not only the spin velocity but the other velocity components in the cyclone.

Because the Reynolds number of the main flow is generally very large, boundary layers in which inertial and viscous stresses balance form on the wall. These may be turbulent, in which case the flow in the main body of the cyclone will be a mean flow, approximately inviscid, on which there will be superimposed small-scale fluctuations. The solution for the essentially inviscid flow in the main body of the cyclone would permit the determination of the particle paths in the cyclone and consequently allow the efficiency to be calculated. The solution would also allow further developments, particularly in the boundary-layer flow regions, to be made.

The results of the analysis for the main flow would be of great benefit if particularly simple functional forms for quantities such as the velocity components could be obtained. This necessarily means that the model will suffer from severe limitations but as long as these are recognized progress can still be made. Earlier work by the present authors (Bloor & Ingham 1973) has shown that the inviscid flow in the cyclone is rotational. The analysis in that case was limited to a small-angled cyclone and furthermore the approach used was a Pohlhausen type of calculation. The form of vorticity distribution used in the analysis was determined by the Pohlhausen method, and originated from the conditions at the inlet, where the fluid was assumed to enter the cyclone with uniform angular momentum.

Another approach to the problem of the flow in the cyclone, retaining the essential feature of a rotational flow, with a non-zero azimuthal component of vorticity, is to consider the generation of what might be called secondary flows. If fluid is introduced into the cyclone with zero azimuthal vorticity, but with angular momentum which varies with radius, i.e. with an axial component of vorticity, then the geometry of the system ensures that an azimuthal component of vorticity is generated. This can be seen by considering the path that must be followed by an element of fluid which enters the cyclone. Geometrical constraints ensure that a radial component of velocity is generated and, indeed, eventually the axial component of velocity may be reversed. This happens in conjunction with the variation in the azimuthal component of velocity that arises due to the conservation of angular momentum.

By considering an entry flow that is purely axial, it is not difficult to see that, in these circumstances, a vortex line, which is initially parallel to the axis of the cyclone, is severely distorted so that components of vorticity in both the radial and azimuthal directions are generated. Intuitively, it can be seen that the azimuthal component of vorticity must have a marked effect on the flow in the axial plane. Furthermore, the induced radial component of vorticity will influence the swirl velocity and thus produce a swirl-velocity distribution that deviates from the usual inviscid free-vortex motion.

As has been mentioned, the nature of the earlier work on the fluid mechanics of a cyclone precluded any direct input of entry conditions. These were determined by what was essentially a consistency argument based on a qualitative knowledge of the form of the flow; this of course is the essence of a Pohlhausen method of solution. In the present approach, more realistic inlet conditions can be modelled. However, it must still be borne in mind that, in practical situations, the flow at entry to the cyclone is three-dimensional and only becomes axially symmetric after the fluid has travelled some distance inside the cyclone. Nevertheless, account can still be taken of the essential feature of the flow at entry, namely, the variation of the angular momentum of the fluid particles. Thus, it is envisaged in this model that at the level in the cyclone where the flow may be regarded as axially symmetric, the fluid has a prescribed angular momentum. It is further assumed that there is no azimuthal component of vorticity due to variations in the total head generated at entry. Thus the components of the velocity at entry, or more precisely, the form of these components at the end of the transition from three-dimensional to axially symmetric flow are prescribed. Consequently, it can be seen that this approach is complementary to the earlier work. Having said this, a rough order-of-magnitude argument produced by the present authors (Bloor & Ingham 1983) indicated that azimuthal vorticity generated by viscous action at entry was unable to account for the level of vorticity evidently present in a cyclone, based on experimental observations.

It will be seen from the comparison of the results of the present analysis with

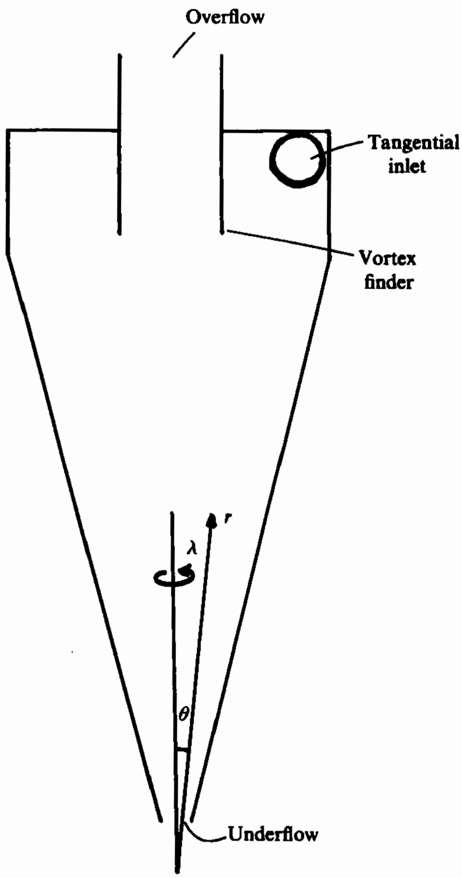

FIGURE 1. A diagram of the cyclone showing the spherical polar coordinate system.

experimental findings that the secondary motions produced from realistic entry conditions are of sufficient strength and do generate the required levels of vorticity in the cyclone.

2. Equations of motion

In figure 1, a diagram of the cyclone is given and the spherical polar coordinate system (r, θ, λ) with the origin at the apex of the cone is also shown. The equation of conservation of mass for an incompressible fluid is then

$$\frac{\partial}{\partial r}(r^2 \sin\theta q_r) + \frac{\partial}{\partial \theta}(r \sin\theta q_\theta) = 0, \tag{1}$$

where the velocity components in the r-, θ- and λ-directions are denoted by q_r, q_θ and q_λ respectively. The fact that the flow is axially symmetric has been used by ensuring that all the derivatives with respect to λ are identically zero.

For steady flow the momentum equation is conveniently written in vector form as

$$\operatorname{grad}\left(\frac{p}{\rho} + \tfrac{1}{2}q^2\right) - \mathbf{q} \times \boldsymbol{\omega} = 0, \tag{2}$$

where $\boldsymbol{q} = (q_r, q_\theta, q_\lambda)$, $\boldsymbol{\omega} = (\omega_r, \omega_\theta, \omega_\lambda)$ is the vorticity, p is the pressure and ρ is the density. In this coordinate system, the vorticity components are defined by

$$\omega_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(q_\lambda \sin\theta), \tag{3a}$$

$$\omega_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(rq_\lambda), \tag{3b}$$

$$\omega_\lambda = \frac{1}{r}\left(\frac{\partial}{\partial r}(rq_\theta)-\frac{\partial q_r}{\partial\theta}\right). \tag{3c}$$

Following Batchelor (1967), it is convenient to introduce the quantity H defined by

$$H = \frac{p}{\rho}+\tfrac{1}{2}q^2, \tag{4}$$

so that H represents the total head of fluid. A stream function ψ is introduced to satisfy the continuity equation (1). Hence ψ is defined by

$$\frac{\partial\psi}{\partial\theta} = r^2\sin\theta q_r, \quad \frac{\partial}{\partial r} = -r\sin\theta q_\theta. \tag{5}$$

Now, taking the scalar product of (2) with a line element parallel to $\boldsymbol{q}$, the term in $\boldsymbol{\omega}$ is eliminated and, using (4) and (5), Bernoulli's equation is obtained in the form

$$\frac{p}{\rho}+\tfrac{1}{2}q^2 = H(\psi). \tag{6}$$

The λ-component of (2) gives, using (3),

$$\left\{q_r\frac{\partial}{\partial r}+\frac{q_\theta}{r}\frac{\partial}{\partial\theta}\right\}(q_\lambda r\sin\theta) = 0. \tag{7}$$

Hence (7) can be integrated to give

$$q_\lambda r\sin\theta = C(\psi), \tag{8}$$

which is simply a statement of the conservation of angular momentum for the fluid. Substituting from this equation for q_λ into ($3a$) gives

$$\omega_r = \frac{1}{r^2\sin\theta}\frac{\mathrm{d}C}{\mathrm{d}\psi}\frac{\partial\psi}{\partial\theta}. \tag{9}$$

Taking the θ-component of (2) and using (5), (8) and (9) to eliminate q_r, q_λ and ω_r respectively, it can be seen that ω_λ is given by

$$\frac{\omega_\lambda}{r\sin\theta} = \frac{1}{r^2\sin^2\theta}C\frac{\mathrm{d}C}{\mathrm{d}\psi}-\frac{\mathrm{d}H}{\mathrm{d}\psi}. \tag{10}$$

In order to determine the form of the functions $C(\psi)$ and $H(\psi)$ it is necessary to consider the boundary conditions to be imposed on the problem. In particular, the way in which the three-dimensional nature of the flow at the inlet is made consistent with the imposed axial symmetry of the problem needs to be examined carefully.

Bloor & Ingham (1973) used a Pohlhausen method to solve the problem and the result of this was that the azimuthal vorticity distribution at entry was predicted rather than prescribed. In effect, the approach assumed that fluid was injected radially into the cyclone with the appropriate azimuthal vorticity distribution and all the fluid particles entering had the same angular momentum. As a result, the quantity $C(\psi)$ was constant and thus the term involving C in (10) was identically zero. The quantity $H(\psi)$ turned out to be proportional to $\psi^{-\frac{5}{3}}$ to leading order. It was recognized at the time that this distribution was unrealistic in certain regions of the flow. Nevertheless, the results gave good overall agreement with experiment and, because of the extremely simple form of the solution, the model was a particularly suitable representation of the basic flow which allowed further study of the boundary layers and of the efficiency to be carried out.

In the present paper, more physically realistic entry conditions will be prescribed. It is assumed that the flow enters the region of the cyclone where the flow is taken to be axially symmetric with a 'top-hat' profile in the velocity component q_λ, and uniform inward velocity perpendicular to the lid of the device, the radial velocity being chosen to ensure that the normal velocity at the wall is zero and that H is a constant. Bearing in mind that the upper part of a cyclone usually has a cylindrical section, these entry conditions seem appropriate.

If the swirl velocity at entry is V, then at entry C is given by

$$C = r\sin\theta V. \tag{11}$$

Further, if the velocity component perpendicular to the lid directed into the cyclone is denoted W, and the radius of the cyclone is R_0, then the stream function at entry is given by

$$\psi = \tfrac{1}{2}W(R_0^2 - r^2\sin^2\theta), \tag{12}$$

where the constant of integration has been chosen so as to make $\psi = 0$ at the cyclone wall. The volume flow rate through the cyclone is Q and is given by

$$Q = \pi W(R_0^2 - R_1^2) \tag{13}$$

where R_1 is defined by the fact that the inlet conditions are equivalent to fluid being injected for $R_1 < r\sin\theta < R_0$ through the 'top' of the cyclone. It is worth noting at this stage that strictly speaking, neither W nor R_1 are known since they depend on the way in which the three-dimensional flow in the cylindrical section of the cyclone develops into the axially symmetric flow. However, they are related to one another through (13), since Q is specified for a particular problem.

Differentiating both (11) and (12) with respect to $r\sin\theta$, it can be seen that

$$C\frac{dC}{d\psi} = -\frac{V^2}{W}. \tag{14}$$

Also, because the radial component of velocity has been chosen to ensure that H is constant at entry,

$$H = W^2 + V^2 + q_r^2 + \frac{p_0}{\rho} = \text{constant}. \tag{15}$$

Using (3) and (5) to express ω_λ in terms of ψ, and substituting from (14) and (15) for C and H into (10), the following partial differential equation for ψ is obtained:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = -\frac{V^2}{W}. \tag{16}$$

It will be found convenient to non-dimensionalize all the physical quantities using the following scales: for lengths R_0 is used, for ψ $Q/2\pi$ is used, and consequently for the velocity components in the axial plane $Q/2\pi R_0^2$ is the appropriate scale. In the following analysis it is understood that all the variables have been non-dimensionalized but, for convenience, will be denoted by their dimensional notation. Equation (16) then becomes

$$\frac{\partial^2\psi}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)=-2\sigma, \tag{17}$$

where

$$\sigma=-\frac{\pi R_0^2\,V^2}{QW}. \tag{18}$$

An examination of (17) shows that there is a solution of the form

$$\psi=r^2F(\theta) \tag{19}$$

and making this substitution yields the following equation for F:

$$2F+\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\sin\theta}\frac{\mathrm{d}F}{\mathrm{d}\theta}\right)=-2\sigma. \tag{20}$$

Clearly, the particular integral is $-\sigma$ and a complementary function is $\sin^2\theta$. Another independent complementary function is of the form $f(\theta)\sin^2\theta$ which on substitution gives

$$f(\theta)=\int^{\theta}\frac{\mathrm{d}t}{\sin^3 t}. \tag{21}$$

Hence a solution for F is of the form

$$F=-\sigma+A\sin^2\theta+B\,(\sin^2\theta\ln(\tan(\tfrac{1}{2}\theta))-\cos\theta), \tag{22}$$

where A and B are constants of integration. In order to obtain a general solution of (20) it is necessary to find the complete set of complementary functions, in other words solutions representing irrotational flow. This was done by Bloor & Ingham (1973) but it was found that, owing to the small values of cone semi-angle α that occur in practice, the corresponding variations in velocities with r did not seem to be present in the experimental data. In view of this it was decided that it would be more useful and instructive to concentrate on the present simple solution although it was recognized that the full solution would be required for more specific boundary conditions.

The boundary conditions required to determine A and B are

$$F(0)=F(\alpha)=0, \tag{23}$$

using the fact that ψ is zero on the wall and on the axis since the possible presence of an air or vapour core has been ignored. Comparison of the theoretical results with experimental data for a cyclone operating without an air core, and also one with an air core, gives reasonable agreement. This suggests that the neglect of the air core in the analysis is not critical, but when an air core is present one must, of course, expect the theory to underestimate the axial velocities between the air core and the vortex finder. Hence A and B can be determined, and using (19) the solution of ψ is given by

$$\psi=\sigma r^2\{[\operatorname{cosec}^2\alpha+\ln(\tfrac{1}{2}\tan\alpha)-\operatorname{cosec}\alpha\cot\alpha]\sin^2\theta-\sin^2\theta\ln(\tfrac{1}{2}\tan\theta)+\cos\theta-1\}, \tag{24}$$

where the term in the square brackets is A/σ.

However, reference to (18) shows that σ is not known, since W is not specified. To determine σ, use is made of the fact that the streamline $\psi = 1$ just enters the vortex finder of the cyclone, the location of which is known to be at $r = r_f$ and $\theta = \theta_f$ say, where in practice θ_f is about $\frac{1}{5}\alpha$. Thus the radius of the vortex finder is $r_f \sin\theta_f$ and σ is determined by

$$1 = \sigma r_f^2\{A \sin^2\theta_f - \sin^2\theta_f \ln(\tfrac{1}{2}\tan\theta_f) + \cos\theta_f - 1\}, \tag{25}$$

using (24).

As was mentioned in the general description of the flow field, the free-vortex motion is modified by the secondary flow, and q_λ can be found by integrating (14). The result is

$$\frac{q_\lambda}{V} = \frac{[1 - Q^2\sigma\psi/(\pi R_0 V)^2]^{\frac{1}{2}}}{r \sin\theta}. \tag{26}$$

Clearly, the second term in the bracket provides the modification to the free-vortex motion represented by the first term. The other two velocity components can be found from (5), and are given by

$$q_r = 2A\cos\theta - 2\sigma[\cos\theta \ln(\tfrac{1}{2}\tan\theta)], \tag{27}$$

$$q_\theta = \frac{2\psi}{\sin 2\theta}. \tag{28}$$

In order to compare the theoretical results with experiment, it is more convenient to resolve these two components into the axial and radial components of velocity, denoted by w and u respectively. Thus u and w are given by

$$u = q_r \sin\theta - q_\theta \cos\theta, \tag{29}$$

$$w = q_r \cos\theta + q_\theta \sin\theta. \tag{30}$$

3. Results and discussion

Before any detailed comparisons with experiment are made, it is useful to show a typical streamline pattern of the flow in the cross-plane. Figure 2 shows such a flow, where the streamlines are plotted for the specified values of ψ/σ. The position of the vortex finder for a particular cyclone geometry determines the value of σ by the condition that the streamline $\psi = 1$ intersects the end of the vortex-finder wall. For example, if the vortex-finder wall is situated at the station X where $\psi/\sigma = 0.06$, then this determines the value of $\sigma (= 1/0.06)$, and the region in which $\psi/\sigma > 0.06$ corresponds to the recirculating zone. It should be noted that this region of the flow field within the cyclone is not covered by the theory. This recirculating flow is driven by the viscous action of the main flow as it shoots rapidly down into the cyclone in a layer adjacent to the wall, not to be confused with a viscous boundary layer, and makes its way to the overflow in a region surrounding the axis. This difficulty is not unexpected in this sort of analysis and was encountered by Long (1956) when examining the flow towards a sink on the axis of a rotating cylinder. To obtain a complete solution to the problem it would be necessary to analyse this region using the general results obtained by Batchelor (1956) and following an analysis based on that of Riley (1981). From a practical point of view, the recirculating zone is of little interest, as it has virtually no direct influence on the separating process. However, the way in which this region might affect the problem is through its influence on the main flow.

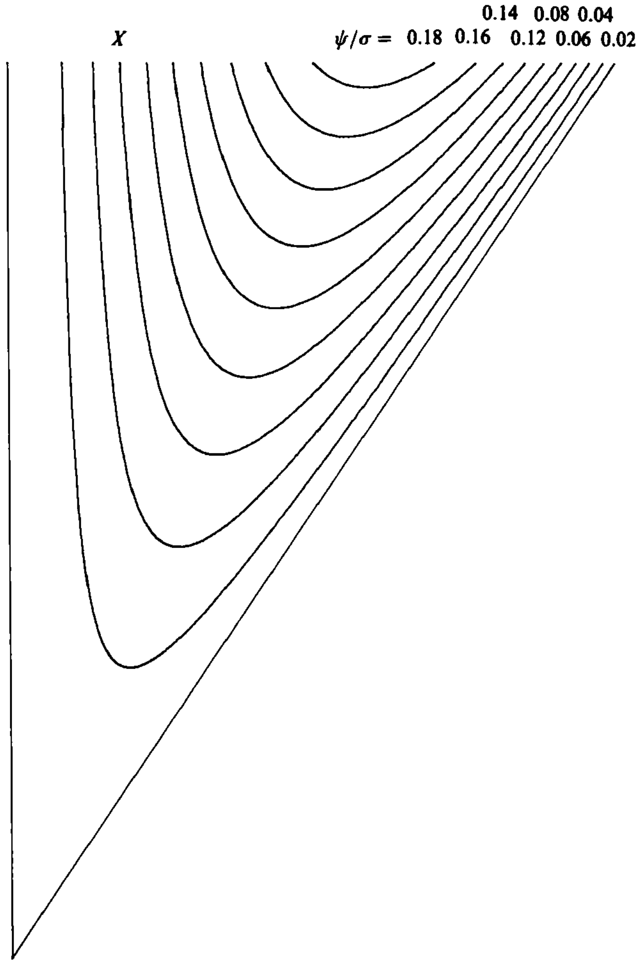

FIGURE 2. Streamline pattern on the axial plane of the cyclone, showing the dependence of σ on the position of the vortex finder since $\psi = 1$ just enters the vortex finder.

The extra complication in analysing this region is enormous; indeed, Riley suggests that it is probably more efficient to solve the complete Navier–Stokes equations numerically. It is therefore fortunate that comparisons of the present theory with experimental results show good agreement, thus indicating that it is legitimate, in the class of problems considered, to neglect this effect. The evident lack of influence of this region on the main flow is probably because, for given inlet conditions, adjustments take place in the three-dimensional flow region to ensure that the flow in the main body of the cyclone satisfies the outlet conditions at the overflow.

In order to compare the theoretical results with the corresponding experimental findings of Kelsall (1952), the parameters in the system are chosen to be those appropriate to Kelsall's apparatus. The axial component of velocity, obtained from (30), is shown in figure 3 at two levels in the cyclone, which correspond to about 0.4 and 0.85 of the distance between the underflow and the vortex finder, above the underflow. These two levels were chosen for comparison as they represented the two extreme levels below the vortex finder in the conical section at which experimental

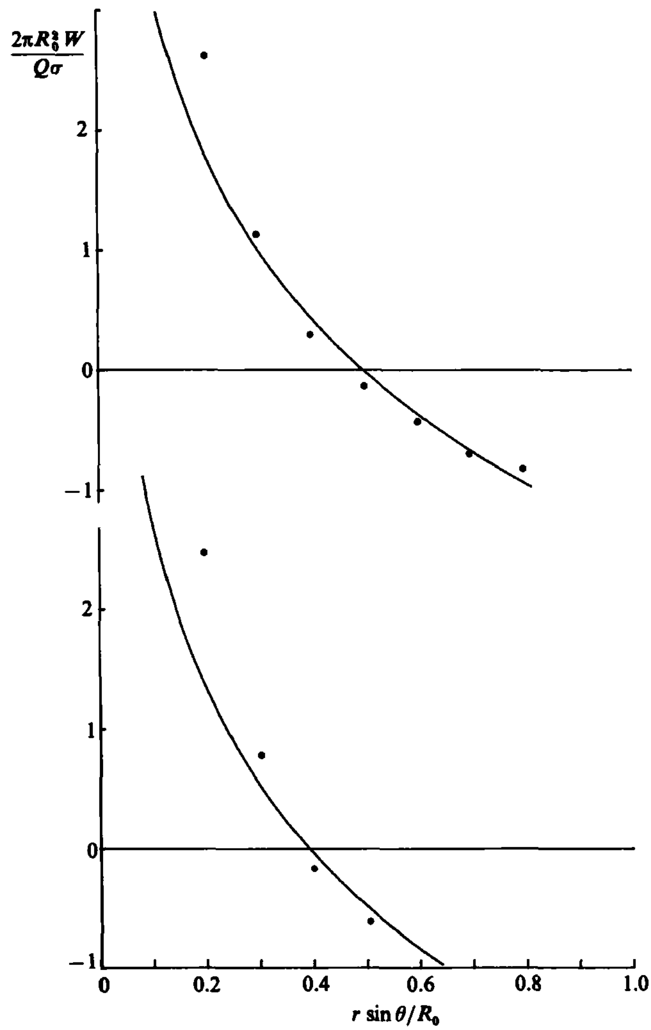

FIGURE 3. The axial velocity component at two levels in the cyclone compared with Kelsall's experimental results (●).

data were available. This is the region where the separation takes place and is therefore the region where it is essential to have a good theoretical model. It can be seen that, on the whole, good agreement is obtained except in the region close to the axis where the experimental values are somewhat larger than those theoretically predicted. This is to be expected, since the presence of the air core surrounding the axis has been neglected in the analysis, and thus no allowance has been made for the reduction in the cross-sectional area of the vortex finder available to the fluid exiting from the cyclone. Kelsall used the continuity equation to derive the radial velocity distribution using the measured axial velocities. These horizontal velocities are, of course, overestimated by the theory for the reason just given. A contributory factor in this discrepancy is the fact that no account has been taken of the short-circuit flow through the boundary layer on the lid of the cyclone, the so called 'leakage effect'. The neglect of the air core in the theory causes an infinite axial velocity to be predicted on the axis of the cyclone. The inclusion of an air core in the analysis results in a

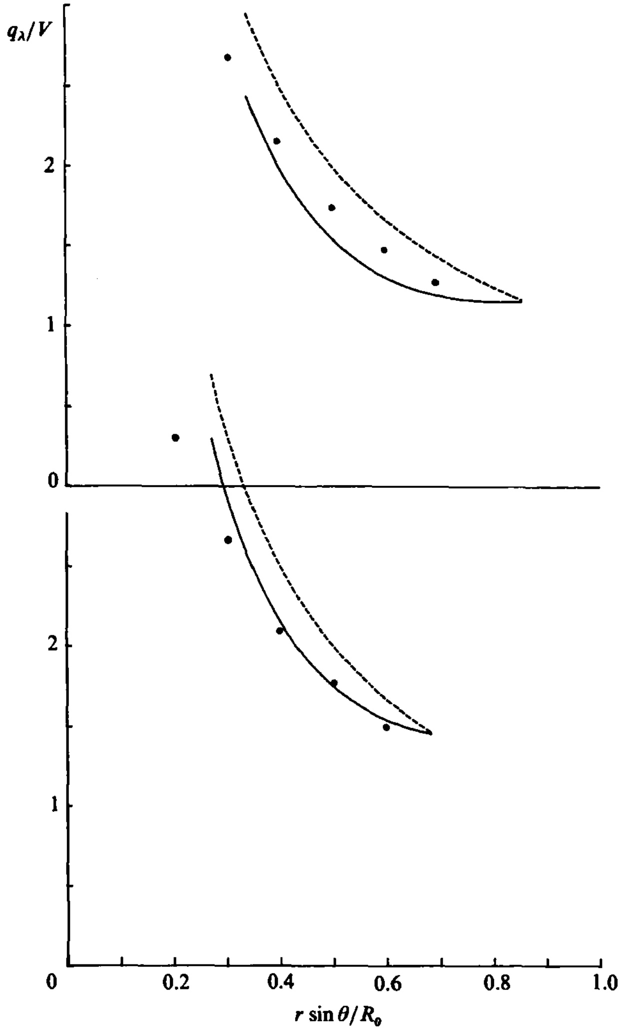

FIGURE 4. The swirl velocity at two levels in the cyclone compared with Kelsall's experimental results (●) and with the appropriate free-vortex motion (----).

formulation of the problem in which no simple analytical solution is possible and a full numerical solution would be required.

It is clear that the general behaviour of the flow is accurately predicted by the theory. In particular, for small-angled cyclones, the solution given by (24) takes a particularly simple form, and is given by

$$\psi = -\sigma r^2\theta^2 \ln\left(\frac{\theta}{\alpha}\right). \tag{31}$$

From this equation, using (5) and (30), the approximate form of the axial velocity component is obtained, namely

$$w = -\sigma\left[(2\theta+2)\ln\left(\frac{\theta}{\alpha}\right)+1\right]. \tag{32}$$

This is approximately zero where $\theta = e^{-\frac{1}{2}}\alpha$, whereas experimentally this is frequently found to be about 0.6α, see e.g. Bradley (1965).

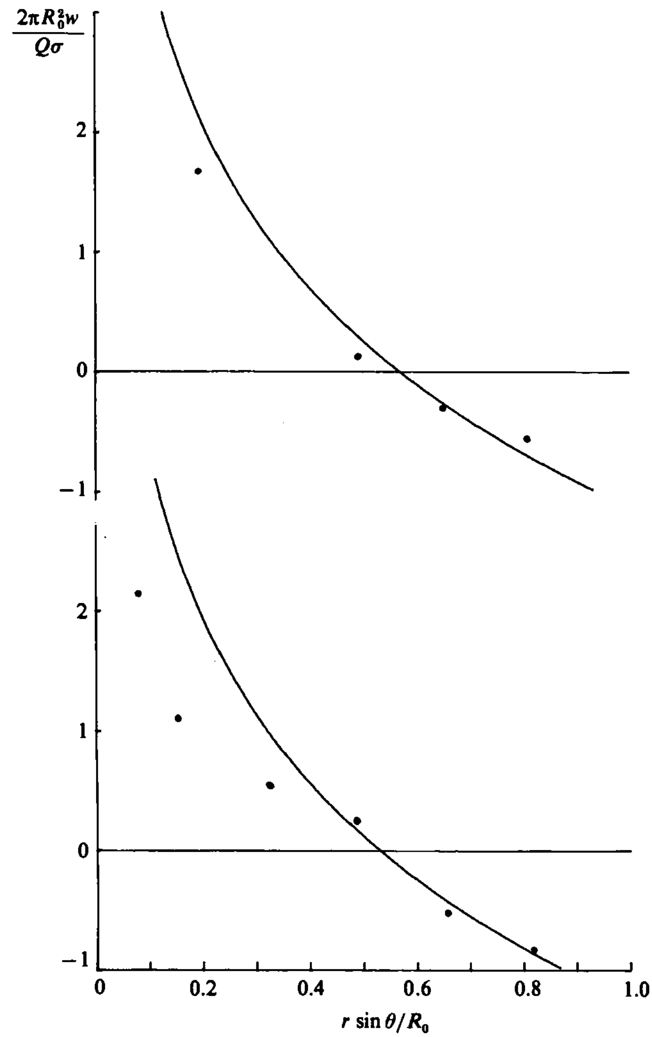

FIGURE 5. The axial velocity component at two levels in the cyclone compared with the experimental results of Knowles *et al.* (●).

The swirl velocity q_λ, obtained from (26), is compared with Kelsall's experimental findings in figure 4. Again, good agreement between theory and experiment is obtained in the region where the effects of viscosity are unimportant. In particular, the deviation of the flow from a free vortex due to the influence of the secondary motions is illustrated by the comparison with the curve $1/r \sin\theta$, which is also plotted.

Recently there have been a number of experimental investigations into the flow in a cyclone which are complementary to the pioneering work of Kelsall. Indeed, one of the criticisms of Kelsall's work was that the vortex finder used, to achieve axially symmetric flow, was unrealistically long. The work of Knowles, Woods & Fuerstein (1973), which measured the velocity components in a cyclone operating without an air core, using high-speed photography of dyed anisole particles, seems an appropriate source for comparison. However, a difficulty in making this comparison is that, in the experiment, a fifth of the total flux through the cyclone was removed at the apex.

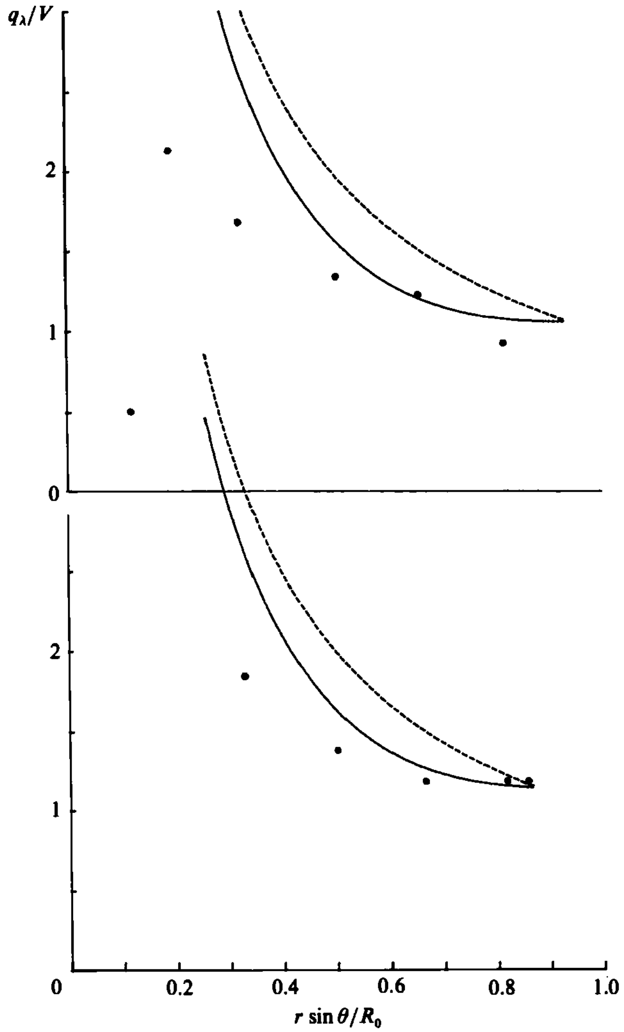

FIGURE 6. The swirl velocity at two levels in the cyclone compared with the experimental results of Knowles *et al.*, shown by (●), and with the appropriate free vortex motion shown (----).

Since the parameter σ is determined by the outlet conditions at the overflow, the value of Q used in the theory for the comparison is taken to be 0.8 times the actual throughput. The remaining parameters in the theory were chosen to match the conditions of the experiment. Figures 5 and 6 show the comparison of the axial and swirl velocity components respectively at levels above the underflow of 0.86 and 0.96 of the length of the conical section. These two levels, although unfortunately close together, nevertheless represent two extreme levels within the conical section at which experimental results were available. Good agreement is again obtained for a system that has markedly different operating conditions and geometry from those of Kelsall. Although the very small radial velocity component was measured in the experiments, the scatter in the results was so great that no worthwhile comparison with the theory could be made.

It can be seen that the present theory gives an adequate representation of the flow field in a cyclone. The very simple form of the solution makes this approach particularly useful in the further development of analyses of various aspects of the flow. The study of viscous effects can be made following the earlier analysis of Bloor & Ingham (1983) and further insight into such features as leakage obtained. It is also clear that the secondary flows induced by the type of entry conditions used in the present analysis, are of sufficient magnitude to produce the distinctive form of the flow in a cyclone. Furthermore, it has been demonstrated that the way in which the entry conditions are modelled has a profound effect on the nature of the flow.

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